Lecture Note
Conformal Mapping
MAT-303(Unit 3)

Dr. B. M. Patel
Government Science College, Gandhinagar
Gujarat University

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Contents

Chapter 1

Conformal Mapping

1.1 Introduction

A complex number $z = x + iy$ can be represented by a point P whose coordinates are (x, y). The axis of x is called real axis and the axis of y is called imaginary axis. The plane is called as a z-plane or a complex plane or Argand plane.

A number of points (x, y) are plotted on z-plane by taking different values of z (i.e. different values of x and y). The curve C is drawn by joining the plotted points in the z-plane.

Now let $w = u + iv = f(z) = f(x + iy)$. To draw a curve of w, we take u-axis and v-axis. By plotting different (u, v) on w-plane and joining them, we get a curve C_1 on w-plane.

For every point (x, y) in the z-plane, the relation $w = f(z)$ defines a corresponding point (u, v) in the w-plane.

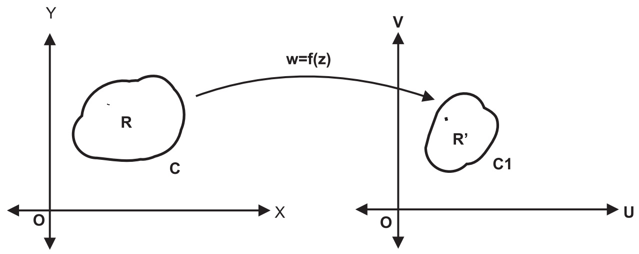

Figure 1.1:

We call this as "transformation" or mapping of z-plane into w-plane. If a point z_0 maps into the point w_0, w_0 is known as the image of z_0.

As the point $P(x, y)$ traces a curve C in z-plane the transformed point $P'(u, v)$ will trace a curve C_1 in w-plane. We say that a curve C in the z-plane is mapped in to the corresponding curve C_1 in w-plane by the relation $w = f(z)$.

1.2 Conformal Transformation

Example 1.2.1. *Transform the curve $x^2 - y^2 = 4$ under the mapping $w = z^2$.*

Solution: $w = z^2 \Rightarrow u + iv = (x + iy)^2 = (x^2 - y^2) + i(2xy)$. This gives

$$u = x^2 - y^2 \quad \text{and} \quad v = 2xy$$

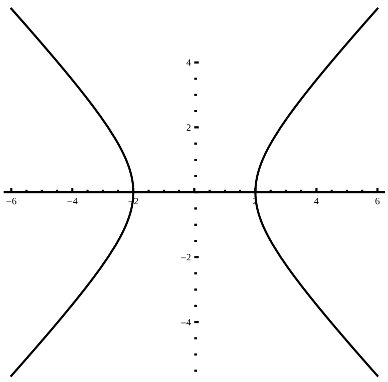

Figure 1.2: z-plane

Image of the curve $x^2 - y^2$ is a straight line, $u = 4$ parallel to the v-axis in w-plane. □

Definition 1.2.1 (Conformal Transformation)**.** Let two curves C_1, C_2 in the z-plane intersect at the point P and the corresponding curve $C_1^{'}, C_2^{'}$ in the z-plane intersect at P'. If the angle of intersection of the curves at P in z-plane is the same as the angle of intersection of the curves of w-plane at P' in magnitude and sense, then the transformation is called *conformal transformation* at P.

Theorem 1.2.2. *Let $f(z)$ be an analytic function of z in a region D of the z-plane and $f'(z) \neq 0$ in D. Then the mapping $w = f(z)$ is conformal at all points of D.*

Proof. Let z_0 be an interior point of the region D and let C_1 and C_2 be two continuous curves passing through z_0. Suppose these curves have definite tangents at z_0 making angles α_1 and α_2 respectively with the the real axis. Take the points z_1 and z_2 on the curves C_1 and C_2 near z_0 at the same distance r from the point z_0. Then we can write

$$z_1 - z_0 = re^{i\theta_1} \quad \text{and} \quad z_2 - z_0 = re^{i\theta_2}$$

As $r \to 0,\ \theta_1 \to \alpha_1$ and $\theta_2 \to \alpha_2$. (see Figure)

Now as a point moves from z_0 to z_1 along C_1, the image point moves along $C_1^{'}$ in the w-plane from w_0 to w_1. Similarly as a point moves from z_0 to z_2 along C_2, the image point moves along $C_2^{'}$ in the w-plane from w_0 to w_2. We suppose that

$$w_1 - w_0 = \rho_1 e^{i\varphi_1} \quad \text{and} \quad w_2 - w_0 = \rho_2 e^{i\varphi_2}$$

Since $f(z)$ is analytic at z_0, we have

$$\lim_{z_1 \to z_0} \frac{f(z_1) - f(z_0)}{z_1 - z_0} = f'(z_0) \quad \Rightarrow \quad \lim_{z_1 \to z_0} \frac{w_1 - w_0}{z_1 - z_0} = f'(z_0) \tag{1.1}$$

As $f'(z_0) \neq 0$, we may write $f'(z_0) = R_0 e^{i\theta_0}$. It follows that

$$\lim_{z_1 \to z_0} \frac{\rho_1 e^{i\varphi_1}}{r e^{i\theta_1}} = R_0 e^{i\theta_0} \quad \Rightarrow \quad \lim_{z_1 \to z_0} \frac{\rho_1}{r_1}\, e^{i(\varphi_1 - \theta_1)} = R_0 e^{i\theta_0}$$

Hence $\lim\limits_{z_1 \to z_0} \varphi_1 - \theta_1 = \theta_0 \quad \Rightarrow \quad \varphi_1 - \alpha_1 = \theta_0 \quad \Rightarrow \quad \varphi_1 = \alpha_1 + \theta_0.$

Thus the curve $C_1^{'}$ has a definite tangent at w_0 making an angle $\alpha_1 + \theta_0$ with the real axis. Similarly it can be shown that the curve $C_2^{'}$ has a definite tangent at w_0 making an angle $\alpha_2 + \theta_0$ with the real axis. Consequently the angle between the tangent at w_0 to the curve $C_1^{'}$ and $C_2^{'}$ is equal to

$$(\alpha_2 + \theta_0) - (\alpha_1 + \theta_0) = \alpha_2 - \alpha_1$$

which is the same as the angle between the tangents to C_1 and C_2 at z_0. Also the angle between the curves has the same sense in the two figures. Hence the mapping $w = f(z)$ is conformal. $\square$

1.3 Some Standard Transformations

1. **Translation:** The mapping is $w = z + c$, where c is a complex constant. Let $z = x + iy, w = u(x, y) + iv(x, y)$ and $c = c_1 + ic_2$. Then $w = z + c$ will imply

$$\begin{aligned} u + iv &= (x + iy) + (c_1 + ic_2) \\ &= (x + c_1) + i(y + c_2) \end{aligned}$$

By comparing real and imaginary parts, we get,

$$u = x + c_1, \text{ and } v = y + c_2$$

Thus, the transformation of a point $P(x, y)$ in the z-plane onto a point $P'(x + c_1, y + c_2)$.

Hence, the transformation is a translation of the axes and preserves the shape and size.

2. **Rotation and Magnification:** This mapping is $w = cz$, where c is a complex constant.

   (a) **Cartesian form:**
   Let $w = u(x, y) + iv(x, y)$, $z = x + iy$ and $c = c_1 + ic_2$. Then $w = cz$ will imply that

   $$\begin{aligned} u + iv &= (c_1 + ic_1)(x + iy) \\ &= (c_1x - c_2y) + i(c_1y + c_2x) \end{aligned}$$

   By comparing real and imaginary parts,

   $$u(x, y) = c_1x - c_2y \text{ and } v(x, y) = c_1y - c_2x.$$

   Thus, the transformations of a point $P(x, y)$ in the z-plane into a point $P'(c_1x - c_2y, c_1y + c_2x)$ in w-plane.

   (b) **Polar form:**
   Let $w = Re^{i\phi}, z = re^{i\theta}$ and $c = \rho e^{i\alpha}$. Then transformation $w = cz$ becomes

   $$Re^{i\phi} = \rho e^{i\alpha} \cdot re^{i\theta} = \rho re^{i(\alpha+\theta)}.$$

   By comparing, we have

   $$R = \rho r \text{ and } \phi = \theta + \alpha$$

   Thus, the transformation maps a point $P(r, \theta)$ in the z-plane into a point $P'(\rho r, \theta + \alpha)$ in the w-plane.
   Hence, the transformations consists of magnification of the radius vector of P by $\rho = |c|$ and its rotation through the angle α.

3. **Inversion and Reflection:** The transformation is $w = \dfrac{1}{z}$.

   (a) **Cartesian Form:** Let $w = u + iv, z = x + iy$, then $w = \dfrac{1}{z}$

   $$\begin{aligned} \Rightarrow u + iv &= \frac{1}{x + iy} \\ &= \frac{1}{x + iy} \times \frac{x - iy}{x - iy} \\ &= \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2} \end{aligned}$$

By comparing the real and the imaginary parts, we get

$$u = \frac{x}{x^2+y^2} \text{ and } v = -\frac{y}{x^2+y^2}$$

Thus, the transformation maps a point $P(x, y)$ in the z-plane into a point $P'\left(\frac{x}{x^2+y^2}, -\frac{y}{x^2+y^2}\right)$ in the w-plane.

(b) **Polar Form:** Let $w = Re^{i\phi}$ and $z = re^{i\theta}$. Then the transformation becomes

$$Re^{i\phi} = \frac{1}{r}e^{-i\theta}$$

so that $R = \frac{1}{r}$ and $\phi = -\theta$.

Thus under the transformation $w = \frac{1}{z}$, any point $P(r, \theta)$ in z-plane is mapped into the point $P'\left(\frac{1}{r}, -\theta\right)$.

Note that, the origin $z = 0$ is mapped to the point $w = \infty$, called the point at infinity.

1.4 Examples

Example 1.4.1. *Consider the transformation $w = z + (1 + i)$ and determine the region in the w-plane corresponding to the triangular region bounded by the lines $x = 0, y = 0$ and $x + y = 1$ in the z-plane.*

Solution: The given triangular region bounded by the lines $x = 0, y = 0$ and $x+y = 1$ is shown in fig. Then the vertices of the triangular region are $(0, 0), (1, 0)$ and $(0, 1)$. Now, the given transformation is

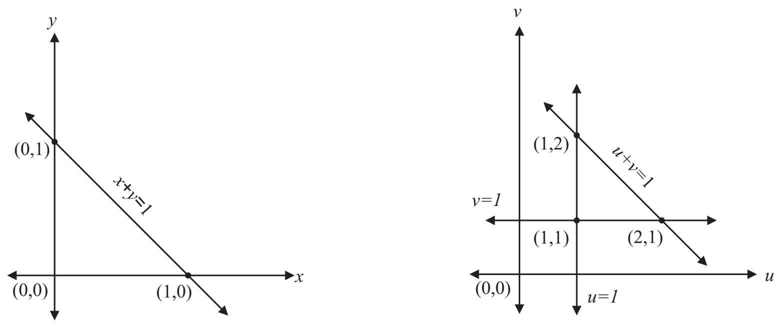

Figure 1.3:

$$\begin{aligned} & w = z + (1+i) \\ \Rightarrow\ & u + iv = (x+iy) + (1+i) \\ \Rightarrow\ & u + iv = (x+1) + i(y+1) \\ \Rightarrow\ & u = x+1 \text{ and } v = y+1. \end{aligned}$$

$x = 0 \Rightarrow u = 1;$ $\quad y = 0 \Rightarrow v = 1;$ $\quad x+y = 1 \Rightarrow u-1+v-1 = 1$

$$\Rightarrow u + v = 3$$

The line $x = 0$ maps into $u = 1$, which is also the vertical line in w-plane.

Also, the line $y = 0$ maps into $v = 1$, which is the horizontal line in w-plane.

And, the line $x + y = 1$ maps into the line $u + v = 3$.

Hence, the region becomes triangle bounded by the lines $u = 1, v = 1, u + v = 3$; which is shown in the fig 1.3. □

Example 1.4.2. *Find the image of the circle $|z| = 2$ under the transformation $w = iz + 1$.*

Solution: Let $w = u + iv$ and $z = x + iy$. Then

$$\begin{aligned} w &= iz + 1 \\ \Rightarrow u + iv &= i(x+iy) + 1 = ix - y + 1 = (1-y) + ix \end{aligned}$$

$$\begin{aligned} \Rightarrow\ & u = 1 - y \text{ and } v = x \\ \Rightarrow\ & y = 1 - u \text{ and } x = v \end{aligned}$$

Now,

$$\begin{aligned} |z| = 2 \ &\Rightarrow\ |z|^2 = 4 \\ &\Rightarrow\ |x+iy|^2 = 4 \\ &\Rightarrow\ x^2 + y^2 = 4 \\ &\Rightarrow\ v^2 + (1-u)^2 = 4 \\ &\Rightarrow\ (u-1)^2 + v^2 = 4 \end{aligned}$$

which is equation of the circle centered at $(1, 0)$ and radius is 2. Thus the transformation rotates the circle by $\frac{\pi}{2}$ and translate its by unity to the right. □

Example 1.4.3. *Find the image of the line $y - x + 1 = 0$ under the mapping $w = \frac{1}{z}$. Show it graphically.*

Solution: Let $w = u + iv, z = x + iy$. Then the transformation $w = \frac{1}{z}$

$$
\begin{aligned}
\Rightarrow \qquad & u + iv = \frac{1}{x + iy} \\
\Rightarrow \qquad & x + iy = \frac{1}{u + iv} \\
\Rightarrow \qquad & x + iy = \frac{1}{u + iv} \times \frac{u - iv}{u - iv} \\
\Rightarrow \qquad & x + iy = \frac{u}{u^2 + v^2} - i\frac{v}{u^2 + v^2} \\
\text{comparing real} \quad & \text{and imaginary parts} \\
\Rightarrow \qquad & x = \frac{u}{u^2 + v^2} \quad y = -\frac{v}{u^2 + v^2}
\end{aligned}
$$

Now

$$
\begin{aligned}
& y - x + 1 = 0 \\
\Rightarrow \ & x - y = 1 \\
\Rightarrow \ & \frac{u}{u^2 + v^2} + \frac{v}{u^2 + v^2} = 1 \\
\Rightarrow \ & u + v = u^2 + v^2 \\
\Rightarrow \ & u^2 - u + v^2 - v = 0 \\
\Rightarrow \ & u^2 - u + \frac{1}{4} + v^2 - v + \frac{1}{4} = \frac{1}{2} \\
\Rightarrow \ & \left(u - \frac{1}{2}\right)^2 + \left(v - \frac{1}{2}\right)^2 = \frac{1}{2}
\end{aligned}
$$

which is the equation of the circle centered at $\left(\frac{1}{2}, \frac{1}{2}\right)$ and radius is $\frac{1}{\sqrt{2}}$ in w-plane.

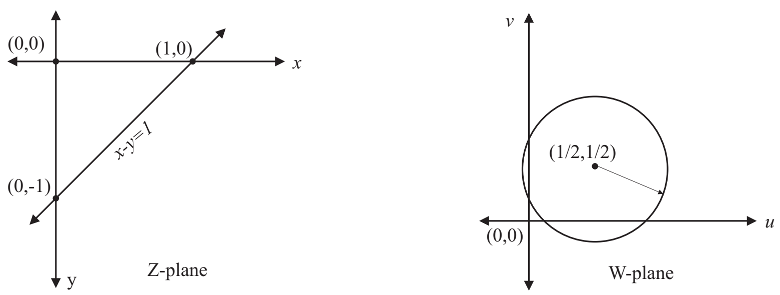

Figure 1.4:

□

Example 1.4.4. *Show that the image of the hyperbola $x^2 - y^2 = 1$ under the transformation $w = \dfrac{1}{z}$ is the lemniscate $R^2 = \cos 2\phi$, where $w = Re^{i\phi}$.*

Solution: Let $w = u + iv, z = x + iy$. Then

$$
\begin{aligned}
& w = \frac{1}{z} \\
\Rightarrow \quad & z = \frac{1}{w} \\
\Rightarrow \quad & x + iy = \frac{1}{u+iv} \times \frac{u-iv}{u-iv} = \frac{u}{u^2+v^2} - i\frac{v}{u^2+v^2} \\
\Rightarrow \quad & x = \frac{u}{u^2+v^2}, \quad y = -\frac{v}{u^2+v^2}
\end{aligned}
$$

Now,

$$
\begin{aligned}
& x^2 - y^2 = 1 \\
\Rightarrow \quad & \left(\frac{u}{u^2+v^2}\right)^2 - \left(\frac{v}{u^2+v^2}\right)^2 = 1 \\
\Rightarrow \quad & u^2 - v^2 = (u^2+v^2)^2
\end{aligned}
$$

putting $u = R\cos\phi$ and $v = R\sin\phi$ ($\because w = Re^{i\phi}$), we get,

$$
\begin{aligned}
R^2(\cos^2\phi - \sin^2\phi) &= \left[R^2(\cos^2\phi + \sin^2\phi)\right]^2 \\
&= R^4 \\
\Rightarrow R^2 &= \cos^2\phi - \sin^2\phi \\
&= \cos 2\phi.
\end{aligned}
$$

Hence, the image of hyperbola in z-plane is $R^2 = \cos 2\phi$, which is lemniscate shown in fig.

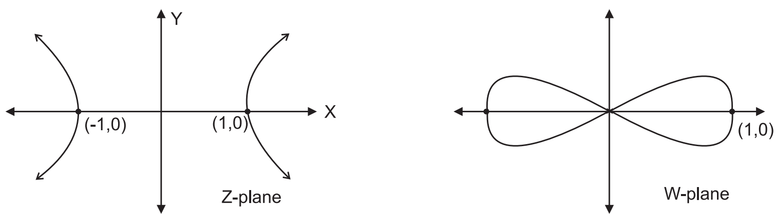

Figure 1.5:

□

Example 1.4.5. *Show that the transformation* $w = \frac{1}{z}$ *maps a circle in* z*-plane to a circle in the* w*-plane or a straight line if the former passes through origin.*

Solution: The general equation of a circle in the z-plane is

$$x^2 + y^2 + 2gx + 2fy + c = 0. \tag{1.2}$$

Let $w = u + iv$ and $z = x + iy$. Then

$$\begin{aligned}
w &= \frac{1}{z} \Rightarrow z = \frac{1}{w} \\
\Rightarrow x + iy &= \frac{1}{u + iv} \\
\Rightarrow x + iy &= \frac{u - iv}{(u + iv)(u - iv)} \\
&= \frac{u}{u^2 + v^2} - i\frac{v}{u^2 + v^2} \\
\Rightarrow x &= \frac{u}{u^2 + v^2} \quad \text{and} \quad y = -\frac{v}{u^2 + v^2}
\end{aligned}$$

Substituting the value of x and y in equation 1.2, we get,

$$\begin{aligned}
&\frac{u^2}{(u^2 + v^2)^2} + \frac{v^2}{(u^2 + v^2)^2} + \frac{2gu}{u^2 + v^2} - \frac{2fv}{u^2 + v^2} + c = 0 \\
\Rightarrow\ &\frac{1}{u^2 + v^2} + \frac{2gu}{u^2 + v^2} - \frac{2fv}{u^2 + v^2} + c = 0 \\
\Rightarrow\ &c(u^2 + v^2) + 2gu - 2fv + 1 = 0 \qquad (1.3)
\end{aligned}$$

Now, if $c \neq 0$ circle does not pass through the origin and equation 1.3 represents a circle in the w-plane.

If $c = 0$ circle passes through the origin and equation 1.3 becomes $2gu - 2fv + 1 = 0$, which is a straight line. □

Example 1.4.6. *Show that the transformation $w = \dfrac{1}{z}$ maps a straight line in z-plane to a straight line in the w-plane or a circle through origin.*

Solution: The general equation of a line in the z-plane is

$$ax + by + c = 0. \qquad (1.4)$$

Let $w = u + iv$ and $z = x + iy$. Then

$$\begin{aligned}
w &= \frac{1}{z} \Rightarrow z = \frac{1}{w} \\
\Rightarrow x + iy &= \frac{1}{u + iv} \\
\Rightarrow x + iy &= \frac{u - iv}{(u + iv)(u - iv)} \\
&= \frac{u}{u^2 + v^2} - i\frac{v}{u^2 + v^2} \\
\Rightarrow x &= \frac{u}{u^2 + v^2},\ y = -\frac{v}{u^2 + v^2}
\end{aligned}$$

Substituting the value of x and y in equation 1.4, we get,

$$\begin{aligned}
&a\frac{u}{(u^2 + v^2)} - b\frac{v}{(u^2 + v^2)} + c = 0 \\
\Rightarrow\ &au - bv + c(u^2 + v^2) = 0 \qquad (1.5)
\end{aligned}$$

Now, if $c \neq 0$ line 1.4 does not passing through the origin and equation 1.5 represents a circle in the w-plane passing through the origin.

If $c = 0$ line 1.4 passes through the origin and equation 1.5 becomes $au - bv = 0$, which is a straight line passing through the origin. □

Example 1.4.7. *Find the image of the circle $x^2 + y^2 = 4y$ under the transformation $w = \frac{1}{z}$. Show it graphically.*

Solution: The given transformation is $w = \frac{1}{z}$. Therefore,

$$z = \frac{1}{w} \quad \Rightarrow \quad x + iy = \frac{1}{u + iv} = \frac{u - iv}{u^2 + v^2}.$$

By comparing real and imaginary parts,

$$x = \frac{u}{u^2 + v^2}, \; y = -\frac{v}{u^2 + v^2}.$$

$x^2 + y^2 = 4y$ or $(x - 0)^2 + (y - 2)^2 = 4$ is the equation of circle centered (0,2) and radius 2, shown in the figure.

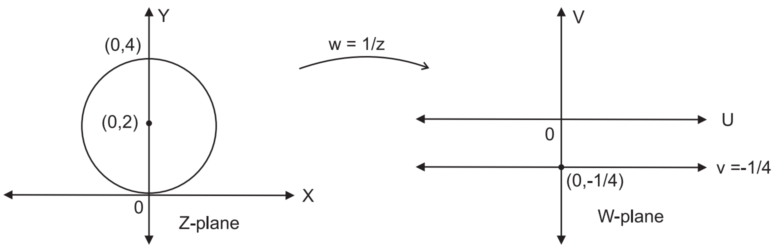

Figure 1.6:

Now,

$$\begin{aligned}
& x^2 + y^2 - 4y = 0 \\
\Rightarrow \quad & \frac{u^2}{(u^2 + v^2)^2} + \frac{v^2}{(u^2 + v^2)^2} + 4\frac{v}{u^2 + v^2} = 0 \\
\Rightarrow \quad & \frac{u^2 + v^2}{(u^2 + v^2)^2} + \frac{4v}{u^2 + v^2} = 0 \\
\Rightarrow \quad & 1 + 4v = 0 \\
\Rightarrow \quad & v = -\frac{1}{4}.
\end{aligned}$$

Which is equation of horizontal line through $(0, -\frac{1}{4})$ in the w-plane. □

Example 1.4.8. *Find the image of the infinite strip $\frac{1}{4} \leq y \leq \frac{1}{2}$ under the transformation $w = \frac{1}{z}$. Also, show the region graphically.*

Solution: Let $w = u + iv, z = x + iy$. Then the transformation $w = \dfrac{1}{z}$

$$\Rightarrow \quad z = \frac{1}{w}$$

$$\Rightarrow \quad x + iy = \frac{1}{u+iv} \times \frac{u-iv}{u-iv} = \frac{u-iv}{u^2+v^2} = \frac{u}{u^2+v^2} - i\frac{v}{u^2+v^2}$$

Thus,

$$x = \frac{u}{u^2+v^2} \tag{1.6}$$

$$y = -\frac{v}{u^2+v^2} \tag{1.7}$$

The strip $\dfrac{1}{4} \leq y \leq \dfrac{1}{2}$ is the z-plane is shown in figure.

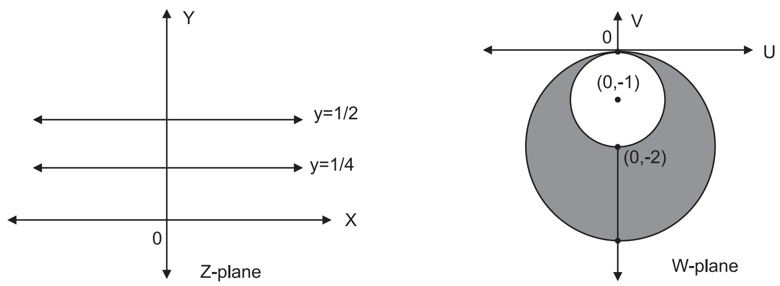

Figure 1.7:

Now,

$$\begin{aligned} y = \frac{1}{4} \qquad &\Rightarrow -\frac{v}{u^2+v^2} = \frac{1}{4} \\ &\Rightarrow -4v = u^2 + v^2 \\ &\Rightarrow 0 = u^2 + v^2 + 4v + 4 - 4 \\ &\Rightarrow u^2 + (v+2)^2 = 4, \end{aligned}$$

which is equation of the circle centered at (0,-2) and radius is 2.

Thus, $y \geq \dfrac{1}{4}$ will imply $u^2 + (v+2)^2 \leq 4$ which is the interior of the circle centered at (0,-2) with the radius is 2.

Similarly,

$$\begin{aligned} y = \frac{1}{2} \qquad &\Rightarrow -\frac{v}{u^2+v^2} = \frac{1}{2} \\ &\Rightarrow -2v = u^2 + v^2 \\ &\Rightarrow 0 = u^2 + v^2 + 2v \\ &\Rightarrow 0 = u^2 + (v+1)^2 - 1 \\ &\Rightarrow u^2 + (v+1)^2 = 1 \end{aligned}$$

which is equation of the circle centered at (0,-1) and radius is 1.

Thus, $y \leq \frac{1}{2}$ will imply $u^2 + (v+1)^2 \geq 1$, which is the exterior of the circle centered at (0,-1) with the radius is 1.

Hence, $\frac{1}{4} \leq y \leq \frac{1}{2} \Rightarrow$ the points exterior to the circle centered at (0,-1) and radius is 1 and interior of the circle centered at (0,-2) and the radius is 2. Which is called the Annulus or Ring in the w-plane. It is shown in the figure. □

Example 1.4.9. *Find the image of the circle $|z + 2i| = 2$ under the transformation $w = \frac{1}{z}$.*

Solution: Let $w = u + iv, z = x + iy$. Now,

$$\begin{aligned}
& |z + 2i| = 2 \\
\Rightarrow \quad & \left|\frac{1}{w} + 2i\right| = 2 \\
\Rightarrow \quad & |1 + 2iw| = 2|w| \\
\Rightarrow \quad & |1 + 2i(u + iv)| = 2|u + iv| \\
\Rightarrow \quad & |1 + 2iu - 2v| = 2|u + iv| \\
\Rightarrow \quad & (1 - 2v)^2 + 4u^2 = 4(u^2 + v^2) \\
\Rightarrow \quad & 1 + 4v^2 - 4v + 4u^2 = 4u^2 + 4v^2 \\
\Rightarrow \quad & 1 - 4v = 0 \\
\Rightarrow \quad & v = \frac{1}{4},
\end{aligned}$$

which is the straight line horizontal to the u-axis in the w-plane. □

Example 1.4.10. *Find the image of the region bounded by the lines $x = 1, y = 1$ and $x + y = 1$ under the transformation $w = z^2$. Show the regions graphically.*

Solution: Let $w = u + iv$ and $z = x + iy$. Then the transformation $w = z^2$ in the cartesian form is

$$u + iv = (x + iy)^2 = x^2 - y^2 + i(2xy)$$

$$u = x^2 - y^2 \tag{1.8}$$

$$v = 2xy \tag{1.9}$$

The triangular region bounded by the lines $x = 1, y = 1, x + y = 1$ is shown in figure The vertices of the triangle are (1,0), (0,1) and (1,1). The corresponding points in the w-plane can be found by using (1.8) and (1.9), which are (1,0), (-1,0) and (0,2) respectively.

For the image of the line $x = 1$ by equation (1.8) and (1.9), we get

$$u = 1 - y^2 \text{ and } v = 2y$$
$$\Rightarrow \quad u = 1 - \left(\frac{v}{2}\right)^2$$
$$\Rightarrow \quad 4 - v^2 = 4u$$
$$\Rightarrow \quad v^2 = -4(u-1)$$

which is the equation of the parabola vertex at (1,0).

Putting $u = 0, v = \pm 2$, i.e., it is the parabola vertex at (1,0) and passes through (0,$\pm$ 2).

Also, for the image of the line $y = 1$ by equation (1.8) and (1.9), we get

$$u = x^2 - 1 \text{ and } v = 2x$$
$$\Rightarrow \quad u = \frac{v^2}{4} - 1$$
$$\Rightarrow \quad 4u = v^2 - 4$$
$$\Rightarrow \quad v^2 = 4(u+1)$$

which is the equation of the parabola in w-plane. Putting $u = 0$, $v = \pm 2$ i.e., it is the parabola having vertex at (-1,0) and passes through
(0, ± 2).

Again, for the image of $x + y = 1$, i.e., $y = 1 - x$ by equation (1.8) and (1.9), we get

$$u = x^2 - (1-x)^2 \text{ and } v = 2x(1-x)$$

$$\begin{aligned} \Rightarrow u &= x^2 - 1 + 2x - x^2 \\ \Rightarrow u &= 2x - 1, \\ \Rightarrow x &= \frac{u+1}{2} \end{aligned}$$

Also, $v = 2x(1-x)$

$$\begin{aligned} \Rightarrow v &= 2\left(\frac{u+1}{2}\right)\left(1 - \frac{u+1}{2}\right) \\ \Rightarrow v &= \frac{1}{2}(u+1)(1-u) = \frac{1}{2}(1-u^2) \\ \Rightarrow \frac{1}{2}u^2 &= -\left(v - \frac{1}{2}\right) \\ \Rightarrow u^2 &= -2\left(v - \frac{1}{2}\right) \end{aligned}$$

which is the equation of the parabola having vertex $\big(0, 1/2\big)$ and passes through the point ($\pm$ 1, 0).

Hence, the image of the triangular region is the region bounded by the parabola $v^2 = -4(u-1)$, $v^2 = 4(u+1)$ and $u^2 = -2\big(v - 1/2\big)$ in the w-plane, which is shown in the figure. $\square$

1.5 Bilinear transformations

A transformation of the form

$$w = f(z) = \frac{az+b}{cz+d},\ ad - bc \neq 0,$$

a, b, c, d are complex constants is called a bilinear transformation or möbius transformation. Bilinear transformation is conformal since

$$\frac{dw}{dz} = \frac{ad-bc}{(cz+d)^2} \neq 0.$$

The inverse mapping of the above transformation is

$$f^{-1}(w) = z = \frac{-dw+b}{cw-a}$$

which is also a bilinear transformations.

We can extend f and f^{-1} to mappings in the extended complex plane. The value $f(\infty)$ should be chosen, so that $f(z)$ has a limit ∞.

Therefore, we define

$$f(\infty) = \lim_{z\to\infty} f(z) = \lim_{z\to\infty} \frac{a+b/z}{c+d/z} = \frac{a}{c}$$

and the inverse is

$$f^{-1}\left(\frac{a}{c}\right) = \infty$$

Similarly, the value $f^{-1}(\infty)$ is obtained by

$$f^{-1}(\infty) = \lim_{w\to\infty} f^{-1}(w) = \lim_{w\to\infty} \frac{-d+b/w}{c-a/w} = \frac{-d}{c}$$

and the inverse is

$$f\left(-\frac{d}{c}\right) = \infty.$$

With these extensions we conclude that the transformation $w = f(z)$ is a one to one mapping of the extended complex z-plane into the extended complex w-plane.

Note 1.5.1.

1. Every bilinear transformation

$$w = \frac{az+b}{cz+d}, ad - bc \neq 0$$

is the combination of basic transformations translations, rotations and magnification and inversion.

2. There exists a unique bilinear transformation that maps four distinct points z_1, z_2, z_3 and z_4 on to four distinct points w_1, w_2, w_3 and w_4 respectively. An implicit formula for the mapping is given by,

$$\frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)} = \frac{(w_1 - w_3)(w_2 - w_4)}{(w_1 - w_4)(w_2 - w_3)}$$

3. The above expression is known as cross-ratio of four points.

4. A point z_0 in complex plane is called a fixed point for the function f if $f(z_0) = z_0$.

Example 1.5.1. *Find the bilinear transformation $w = f(z)$. Which maps the points $z = 1, i, -1$ onto the points $w = i, 0, -i$. Hence find the image of $|z| \leq 1$, interior of the circle centered at the origin and radius 1.*

Solution: Let $z_1 = 1, z_2 = i, z_3 = -1$ and $z_4 = z$ and corresponding images be $w_1 = i, w_2 = 0, w_3 = -i$ and $w_4 = w$.

Now, we know that the cross-ratio of four points is invariant under a bilinear transformation.

$$\begin{aligned}
& \frac{(w_1 - w_3)(w_2 - w_4)}{(w_1 - w_4)(w_2 - w_3)} = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)} \\
\Rightarrow\; & \frac{(i + i)(0 - w)}{(i - w)(0 + i)} = \frac{(1 + 1)(i - z)}{(1 - z)(i + 1)} \\
\Rightarrow\; & \frac{-2iw}{(i - w)i} = \frac{2(i - z)}{(1 + i)(1 - z)} \\
\Rightarrow\; & \frac{w}{w - i} = \frac{z - i}{(1 + i)(z - 1)} \\
\Rightarrow\; & w(1 + i)(z - 1) = (z - i)(w - i) \\
\Rightarrow\; & w\Big[(z - 1) + i(z - 1)\Big] = (z - i)w - i(z - i) \\
\Rightarrow\; & w\Big[(z - 1) + i(z - 1) - (z - i)\Big] = -i(z - i) \\
\Rightarrow\; & w\Big[(z - 1) + i(z - 1) - z + i\Big] = -i(z - i) \\
\Rightarrow\; & w\Big[i(z - 1) - 1 + i\Big] = -iz - 1 \\
\Rightarrow\; & w(iz - 1) = -(1 + iz) \\
\Rightarrow\; & w = \frac{1 + iz}{1 - iz}
\end{aligned}$$

Thus, $w = \dfrac{1+iz}{1-iz}$ is the required bilinear transformation. Also,

$$\begin{aligned}
& w(1-iz) = 1+iz \\
\Rightarrow\quad & w-1 = (w+1)iz \\
\Rightarrow\quad & z = \frac{w-1}{i(w+1)} \\
\Rightarrow\quad & z = -\frac{i(w-1)}{w+1} \\
\Rightarrow\quad & z = \frac{i(1-w)}{1+w}
\end{aligned}$$

Now, for the image of $|z| \leq 1$

$$\begin{aligned}
\Rightarrow\quad & |i\frac{1-w}{1+w}| \leq 1 \\
\Rightarrow\quad & |i||\frac{1-w}{1+w}| \leq 1 \\
\Rightarrow\quad & |1-w| \leq |1+w| \\
\Rightarrow\quad & |1-w|^2 \leq |1+w|^2 \\
\Rightarrow\quad & |1-u-iv|^2 \leq |1+u+iv|^2 \quad (\because w = u+iv) \\
\Rightarrow\quad & (1-u)^2 + v^2 \leq (1+u)^2 + v^2 \\
\Rightarrow\quad & 1 - 2u + u^2 + v^2 \leq 1 + 2u + u^2 + v^2 \\
\Rightarrow\quad & 4u \geq 0 \\
\Rightarrow\quad & u \geq 0
\end{aligned}$$

which is half of the w-plane includes the first and the fourth quadrant. □

Example 1.5.2. *Find the image of the circle $|z| = 1$ in the w-plane under the bilinear transformation $w = f(z) = \dfrac{z-i}{1-iz}$. Also, find the fixed points of f.*

Solution: Here the transformation is

$$\begin{aligned}
& w = \frac{z-i}{1-iz} \\
\Rightarrow\quad & w(1-iz) = z-i \\
\Rightarrow\quad & w+i = (iw+1)z \\
\Rightarrow\quad & z = \frac{w+i}{1+iw}
\end{aligned}$$

To find the image of the circle $|z| = 1$,

$$\begin{aligned}
& |z| = 1 \\
\Rightarrow \quad & \left|\frac{w+i}{1+iw}\right| = 1 \\
\Rightarrow \quad & |w+i| = |1+iw| \\
\Rightarrow \quad & |u+iv+i| = |1+i(u+iv)| \quad (\because w = u+iv) \\
\Rightarrow \quad & |u+i(v+1)|^2 = |1-v+iu|^2 \\
\Rightarrow \quad & u^2+(v+1)^2 = (1-v)^2+u^2 \\
\Rightarrow \quad & u^2+v^2+2v+1 = 1+u^2+v^2-2v \\
\Rightarrow \quad & 4v = 0 \\
\Rightarrow \quad & v = 0
\end{aligned}$$

which is the equation of the u-axis (or the real axis) of the w-plane.

For fixed points of f, let $f(z) = z$ then

$$\begin{aligned}
& \frac{z-i}{1-iz} = z \\
\Rightarrow \quad & z-i = z-iz^2 \\
\Rightarrow \quad & z^2 = 1 \\
\Rightarrow \quad & z = \pm 1.
\end{aligned}$$

Hence $z = 1$ and $z = -1$ are the fixed points of f. □

Example 1.5.3. *If w_1, w_2, w_3, w_4 are distinct images of z_1, z_2, z_3, z_4 (all distinct) under the transformation, $w = \dfrac{az+b}{cz+d}$ $(ad-bc \neq 0)$. Then show that*

$$\frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)} = \frac{(w_1-w_2)(w_3-w_4)}{(w_1-w_4)(w_3-w_2)}.$$

Solution: We have, w_1, w_2, w_3, w_4 are distinct images of z_1, z_2, z_3, z_4 respectively under the transformation $w = \dfrac{az+b}{cz+d}$ $(ad-bc \neq 0)$. So

$$w_i = \frac{az_i+b}{cz_i+d} \quad (ad-bc \neq 0) \quad (i = 1,2,3,4.) \tag{1.10}$$

Now,

$$\begin{aligned}
w_1 - w_2 &= \frac{az_1+b}{cz_1+d} - \frac{az_2+b}{cz_2+d} \\
&= \frac{(az_1+b)(az_2+b)-(cz_1+d)(cz_2+d)}{(cz_1+d)(cz_2+d)} \\
&= \frac{(ad-bc)(z_1-z_2)}{(cz_1+d)(cz_2+d)} \qquad (1.11)
\end{aligned}$$

From (1.11) we have

$$\begin{aligned}\frac{(w_1-w_2)(w_3-w_4)}{(w_1-w_4)(w_3-w_2)} &= \frac{\left[\frac{(ad-bc)(z_1-z_2)}{(cz_1+d)(cz_2+d)}\right]\left[\frac{(ad-bc)(z_3-z_4)}{(cz_3+d)(cz_4+d)}\right]}{\left[\frac{(ad-bc)(z_1-z_4)}{(cz_1+d)(cz_4+d)}\right]\left[\frac{(ad-bc)(z_3-z_2)}{(cz_2+d)(cz_2+d)}\right]} \\ &= \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)}\end{aligned}$$

Example 1.5.4. *Find the bilinear transformation which maps the points $z = 1, i, -1$ onto the points $w = 0, 1, \infty$.*

Solution: Let $z_1 = 1, z_2 = i, z_3 = -1$ and $w_1 = 0, w_2 = 1, w_3 = \infty$.

By definition, the bilinear transformation is,

$$w = \frac{az+b}{cz+d}, \qquad ad - bc \neq 0 \tag{1.12}$$

where a, b, c, d are complex numbers.

Since the images of z_1, z_1 and z_3 are w_1, w_2 and w_3 respectively. The points z_1, z_2, z_3 and w_1, w_2, w_3 satisfy the given equation 1.12.

Therefore, we have

$$0 = \frac{a(1)+b}{c(1)+d} \quad 1 = \frac{a(i)+b}{c(i)+d} \quad \text{and} \quad \infty = \frac{a(-1)+b}{c(-1)+d}$$

On simplification, we get,

$$a + b = 0 \tag{1.13}$$

$$ai + b = ci + d \tag{1.14}$$

$$-c + d = 0 \tag{1.15}$$

From equation 1.13 and 1.15, we get, $b = -a$ and $d = c$. Now, equation 1.14 becomes,

$$\begin{aligned} & ai + b = ci + d \\ \Rightarrow \quad & ai - a = ci + c \\ \Rightarrow \quad & a(-1+i) = c(1+i) \\ \Rightarrow \quad & c = \left(\frac{-1+i}{1+i}\right)a \\ \Rightarrow \quad & c = \left[\frac{(-1+i)(1-i)}{1-i^2}\right]a \\ \Rightarrow \quad & c = \left[-\frac{(1-2i+i^2)}{2}\right]a \\ \Rightarrow \quad & c = ia. \end{aligned}$$

Thus, $b = -a, c = d = ia$.

Now, from equation (1.12), we get,

$$\begin{aligned} w &= \frac{az+b}{cz+d} = \frac{az-a}{iaz+ia} = \frac{a(z-1)}{ia(z+1)} = \frac{i(z-1)}{i^2(z+1)} \quad (a \neq 0) \\ &= -\frac{i(z-1)}{(z+1)} = \frac{i(1-z)}{(z+1)}. \end{aligned}$$

Hence, $w = \dfrac{i(1-z)}{1+z}$ is the required bilinear transformation. □

Example 1.5.5 (***Example 1.5.4 by alternate method***)**.** *Find the bilinear transformation which maps the point $z = 1, i, -1$ on to the points $w = 0, 1, \infty$.*

Solution: Let $z_1 = 1, z_2 = i, z_3 = -1, z_4 = z$ and $w_1 = 0, w_2 = 1, w_3 = \infty, w_4 = w$.

Now we know that the cross ratio for the bilinear transformation is constant.

$$\begin{aligned} & \frac{(w_1-w_2)(w_3-w_4)}{(w_1-w_4)(w_3-w_2)} = \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)} \\ \Rightarrow\ & \frac{(w_1-w_2)(1-\frac{w_4}{w_3})}{(w_1-w_4)(1-\frac{w_2}{w_3})} = \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)} \\ \Rightarrow\ & \frac{(0-1)(1-0)}{(0-w)(1-0)} = \frac{(1-i)(-1-z)}{(1-z)(-1-i)} \\ \Rightarrow\ & \frac{-1}{-w} = \frac{(1-i)(1+z)}{(1-z)(1+i)} \\ \Rightarrow\ & \frac{1}{w} = \frac{(1-i)^2(1+z)}{(1-z)(1+1)} \\ \Rightarrow\ & \frac{1}{w} = \frac{(1-2i-1)(1+z)}{2(1-z)} \\ \Rightarrow\ & \frac{1}{w} = \frac{-2i(1+z)}{2(1-z)} \\ \Rightarrow\ & \frac{1}{w} = \frac{i(z+1)}{(z-1)} \\ \Rightarrow\ & w = \frac{(z-1)}{i(z+1)} \\ \Rightarrow\ & w = \frac{i(1-z)}{(1+z)} \end{aligned}$$

Hence, $w = \dfrac{i(1-z)}{1+z}$ is the required bilinear transformation. □

Example 1.5.6. *Show that the condition for the transformation $w = \dfrac{az+b}{cz+d}$ to make the circle $|w| = 1$ correspond to a straight line in the z-plane is $|a| = |c|$.*

Solution: $w = \dfrac{az+b}{cz+d}$, $a, b, c, d \in \mathbb{C}$. Let $a = a_1 + ia_2$, $b = b_1 + ib_2$, $c = c_1 + ic_2$, $d = d_1 + id_2$, where $a_j, b_j, c_j, d_j \in \mathbb{R}$, $j = 1, 2$.

$$\begin{aligned} w &= \frac{az+b}{cz+d} \\ &= \frac{(a_1+ia_2)(x+iy)+(b_1+ib_2)}{(c_1+ic_2)(x+iy)+(d_1+id_2)} \\ &= \frac{(a_1x-a_2y+b_1)+i(a_2x+a_1y+b_2)}{(c_1x-c_2y+d_1)+i(c_2x+c_1y+d_2)} \end{aligned}$$

Now,

$$\begin{aligned} & |w| = 1 \\ \Rightarrow\ & |w|^2 = 1 \\ \Rightarrow\ & \left|\frac{(a_1x-a_2y+b_1)+i(a_2x+a_1y+b_2)}{(c_1x-c_2y+d_1)+i(c_2x+c_1y+d_2)}\right|^2 = 1 \\ \Rightarrow\ & \Big|(a_1x-a_2y+b_1)+i(a_2x+a_1y+b_2)\Big|^2 = \Big|(c_1x-c_2y+d_1)+i(c_2x+c_1y+d_2)\Big|^2 \\ \Rightarrow\ & (a_1x-a_2y+b_1)^2+(a_2x+a_1y+b_2)^2 = (c_1x-c_2y+d_1)^2+(c_2x+c_1y+d_2)^2 \\ \Rightarrow\ & (a_1^2+a_2^2-c_1^2-c_2^2)(x^2+y^2)+2(a_1b_1+a_2b_2+c_1d_1+c_2d_2)y+(b_1^2+b_2^2+d_1^2+d_2^2) = 0 \end{aligned}$$

Now, if the circle $|w| = 1$ corresponds to a straight line in the z-plane then

$$\begin{aligned} & (a_1^2+a_2^2-c_1^2-c_2^2) = 0 \\ \Rightarrow\ & a_1^2+a_2^2 = c_1^2+c_2^2 \\ \Rightarrow\ & |a_1+ia_2|^2 = |c_1+ic_2|^2 \\ \Rightarrow\ & |a_1+ia_2| = |c_1+ic_2| \\ \Rightarrow\ & |a| = |c|. \end{aligned}$$

□

Example 1.5.7. *Show that the transformation* $w = \dfrac{i(1-z)}{1+z}$ *maps the circle* $|z| = 1$ *into the real axis of the* w*-plane and the interior of the circle* $|z| < 1$ *into the upper half of the* w*-plane.*

Solution: $w = \dfrac{i(1-z)}{1+z}$. Then $z = \dfrac{-(w-i)}{w+i}$

$$|z| = \left|\frac{w-i}{w+i}\right| = \left|\frac{u+i(v-1)}{u+i(v+1)}\right| \tag{1.16}$$

Now,

$$\begin{aligned}
& |z| = 1 \\
\Rightarrow\ & \left|\frac{u+i(v-1)}{u+i(v+1)}\right| = 1 \\
\Rightarrow\ & |u+i(v-1)| = |u+i(v+1)| \\
\Rightarrow\ & u^2+(v-1)^2 = u^2+(v+1)^2 \qquad (1.17) \\
\Rightarrow\ & u^2+v^2-2v+1 = u^2+v^2+2v+1 \\
\Rightarrow\ & -2v = 2v \\
\Rightarrow\ & 4v = 0 \\
\Rightarrow\ & v = 0
\end{aligned}$$

which is the real axis of the w-plane.

If $|z| < 1$ then from (1.17), we get

$$\begin{aligned}
& u^2+(v-1)^2 < u^2+(v+1)^2 \\
\Rightarrow\ & (v-1)^2 < (v+1)^2 \\
\Rightarrow\ & v^2-2v+1 < v^2+2v+1 \\
\Rightarrow\ & 0 < 4v \\
\Rightarrow\ & v > 0
\end{aligned}$$

which is the upper half of the w-plane. □

Example 1.5.8. *Find the image of the first quadrant of z-plane under the transformation $w = z^2$.*

Solution: Let $w = Re^{i\phi}, z = re^{i\theta}$. Then the transformation $w = z^2$ becomes $Re^{i\phi} = r^2e^{2i\theta}$

$$\Rightarrow R = r^2 \text{ and } \phi = 2\theta.$$

Now, the first quadrant in the z-plane, can be expressed in polar form as $0 \leq r < \infty$ and $0 \leq \theta \leq \pi/2$.
$\therefore 0 \leq r^2 < \infty$ and $0 \leq 2\theta \leq \pi$.
Thus, $0 \leq R < \infty$ and $0 \leq \phi \leq \pi$ which is upper half of the w-plane. □